

DEPARTMENT OF MATHEMATICS
DREXEL UNIVERSITY
PH.D. QUALIFYING EXAMINATION
JUNE 15, 2011

Instructions:

- The exam consists of six problems which are equally weighted. The time for examination is **4** hours.
 - Do **4** out of 6 of the analysis questions in Section 1.
 - Do **2** out of 3 of the linear algebra questions in Section 2.
 - Indicate clearly which of your questions are to be graded. If you do **not** indicate which of your questions are to be graded, the default will be to grade questions one through four of the analysis section and questions one and two of the linear algebra section.
 - Please ask the proctor about any obvious typographic errors.
 - Along with this list of problems, you will be given two examination notebooks. Use one of them for presenting your solutions. The other one may be used for auxiliary calculations. Both notebooks must be submitted when the exam is over.
 - Every solution should be given a concise but sufficient explanation and written up legibly. Try to keep a one inch margin on the papers.
 - This is a closed book exam.
 - No electronic devices are allowed.
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Remember: you are to answer 4 out of the following 6 Analysis problems.

- (1) (a) Let (X, d) be a metric space, such that X is a finite set. Let $V \subseteq X$ be given. Show that V is an open set.
 (b) Let $U \subseteq \mathbb{R}$ be an open set. Show that U can be written as an at most countable union of open intervals.

- (2) (a) Let a_k be a sequence such that $a_k > k!$ for all $k \in \mathbb{N}$. Let $f(x) = \sum_{k=0}^{\infty} \frac{x^k}{a_k}$. Prove that f is integrable

on any interval $[c, d]$.

(b) Recall that a function $f : \mathbb{R} \rightarrow \mathbb{R}$ is Lipschitz continuous if there exists a constant $c > 0$ such that for all $x \in \mathbb{R}$ and for all $y \in \mathbb{R}$, $|f(x) - f(y)| < c|x - y|$. Show that $g(x) = e^{-|x|}$ is Lipschitz continuous.

- (3) Let f be a real-valued integrable function on $[0, 1]$.

(a) Find the pointwise limit of

$$\frac{1}{n} \ln(1 + e^{nf(x)}).$$

(b) Find the limit of the integrals:

$$\lim_{n \rightarrow \infty} \frac{1}{n} \int_0^1 \ln(1 + e^{nf(x)}) dx.$$

Justify all your steps.

- (4) (a) Let X be the subset of $C([0, 1])$ consisting of all functions ϕ with continuous first derivative, $\phi(0) = 0$, and

$$\int_0^1 |\phi'(x)| dx \leq 1.$$

Is X equicontinuous? Prove your answer.

(b) Suppose that f is continuous for $x \geq 0$, differentiable for $x > 0$, $f(0) = 0$ and the derivative f' is monotonically increasing. Define $g(x) = f(x)/x$ for $x > 0$. Prove that g is monotonically increasing.

- (5) (a) Let P denote the Cantor set. Let $C(P)$ denote the metric space of all continuous functions on P with the metric $d(f, g) = \max\{|f(x) - g(x)| : x \in P\}$ where $f, g \in C(P)$. Show that $C(P)$ is a separable metric space by exhibiting an explicit countable dense subset D of $C(P)$ and proving that D is dense.

(b) Let $f_n(x) = \sqrt{x^2 + 1/n}$. Show that the sequence $\{f_n(x)\}$ converges uniformly to $f(x) = |x|$ on the entire real line \mathbb{R} .

- (6) Let $F(x, y, z) = (x^2 + z^2 - 4)^2 + y - 16$.

(a) Find explicitly the set of all points $(x_0, y_0, z_0) \in \mathbb{R}^3$ at which it is **not** possible to use the Implicit Function Theorem in order to find a local implicit function $z = f(x, y)$ so that $F(x, y, f(x, y)) = 0$.

(b) Find a point $(x_1, y_1, z_1) \in \mathbb{R}^3$, an **explicit** neighborhood U of the form $(a, b) \times (c, d)$ of (x_1, y_1) in \mathbb{R}^2 , and an **explicit** function $f(x, y) : U \rightarrow \mathbb{R}$ such that the following condition holds: the function f satisfies the equation $F(x, y, f(x, y)) = 0$ for all $(x, y) \in U$.

2. LINEAR ALGEBRA

Remember: you are to answer 2 out of the following 3 Linear Algebra problems.

- (1) (a) Find the singular value decomposition of the matrix

$$\begin{bmatrix} 1 & -3 \\ 3 & -1 \end{bmatrix}.$$

- (b) Find the Jordan canonical form of the matrix

$$M = \begin{bmatrix} -7 & 3 & -5 \\ 7 & -1 & 5 \\ 17 & -6 & 12 \end{bmatrix}.$$

- (2) (a) Let $A, B \in \mathbb{C}^{n \times n}$ be such that $(AB)^n = 0$. Do we have $(BA)^n = 0$? Provide a proof if the answer is yes, or a counterexample if the answer is no.
(b) True or false: if $A, B \in \mathbb{C}^{n \times n}$ satisfy $AB = BA$, then they have a common eigenvector; $A, B \in \mathbb{C}^{n \times n}$ satisfy $AB = BA$, then they have a common eigenvalue. Justify your answers.
- (3) (a) Prove that the only nilpotent and normal matrix is the zero matrix.
(b) Prove that if two matrices $A, B \in \mathbb{C}^{n \times n}$ are unitarily equivalent, then

$$\sum_{i,j=1}^n |a_{i,j}|^2 = \sum_{i,j=1}^n |b_{i,j}|^2.$$