Department of Mathematics
Drexel University
Ph.D. Qualifying Examination
June 19, 2009

Instructions:

- The exam consists of six problems which are equally weighted.
  The time for examination is $\mathbf{3\frac{1}{2}}$ hours.
- Do **4** out of 6 of the analysis questions in Section 1.
- Do **2** out of 3 of the linear algebra questions in Section 2.
- Indicate clearly which of your questions are to be graded. If you do **not** indicate which of your questions are to be graded, the default will be to grade questions one through four of the analysis section and questions one and two of the linear algebra section.
- Please ask the proctor about any obvious typographic errors.
- Along with this list of problems, you will be given two examination notebooks. Use one of them for presenting your solutions. The other one may be used for auxiliary calculations. Both notebooks must be submitted when the exam is over.
- Every solution should be given a concise but sufficient explanation and written up legibly. Try to keep a one inch margin on the papers.
- This is a closed book exam.
- No electronic devices are allowed.

Remember: you are to answer 4 out of the following 6 Analysis problems.

(1) (a) Let $h : [0,1) \to \mathbb{R}$ be a real-valued function defined on the half-open interval $[0,1)$. Prove that if h is uniformly continuous, there exists a unique continuous function $g : [0,1] \to \mathbb{R}$ such that $g(x) = h(x)$ for all $x \in [0,1)$.
(b) Consider the function $f : [0,1] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 0, & x \text{ irrational}, \\ 1/q, & x = p/q > 0 \text{ lowest terms}, \\ 0, & x = 0. \end{cases}$$

Show that f is continuous at $x = 0$ but does *not* have a (right-hand) derivative at $x = 0$.

(2) (a) Let $f : \mathbb{R}^2 \to \mathbb{R}$ have continuous partial derivatives. Suppose there exists a positive constant K such that

$$\left| \frac{\partial f}{\partial x_j}(x) \right| \leq K,$$

for $1 \leq j \leq 2$ and for all $x = (x_1, x_2) \in \mathbb{R}^2$. Prove the following inequality:

$$|f(x) - f(y)| \leq \sqrt{2} K \, \|x - y\|$$

where $\|u\| = \|(u_1, u_2)\| = \sqrt{u_1^2 + u_2^2}$.
(b) Let E be a subset of $\mathbb{R}^n$. Let E' denote the set of all limit points of E. Prove that E' is a closed set. Note: a point x is a *limit point* of E if every open set that contains x also contains a point of E distinct from x.

(3) (a) Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable. Show that if the differential $T'(x)$ is invertible for all $x \in \mathbb{R}^n$, then the image of any open subset of $\mathbb{R}^n$ under T is also open.
(b) Show that there exists an $\varepsilon > 0$ such that for any 2×2 real matrix A satisfying $|a_{ij}| < \varepsilon$ for all entries of a_{ij} of A, there exists a real 2×2 real matrix X satisfying

$$X^2 + X^T = A$$

where X^T is the transpose of X. Is X unique?

(4) (a) Let a_n be a sequence of real numbers which converges to the limit a. Define $s_n = \dfrac{1}{n} \displaystyle\sum_{k=1}^{n} a_k$. Show that s_n also converges to a.
(b) Let $f(x)$ be a differentiable function on the closed interval $[x_1, x_2]$ where $0 < x_1 < x_2$. Prove that there exists $\xi \in (x_1, x_2)$ such that

$$\frac{1}{x_2 - x_1} \begin{vmatrix} f(x_1) & f(x_2) \\ x_1 & x_2 \end{vmatrix} = f(\xi) - \xi f'(\xi).$$

(5) (a) For any positive integer n, let the function $\phi_n : \mathbb{R} \to \mathbb{R}$ be defined by

$$\phi_n(x) = \begin{cases} n, & x \in [-\frac{1}{2n}, \frac{1}{2n}], \\ 0, & \text{otherwise}. \end{cases}$$

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Show that

$$\lim_{n \to \infty} \int_{\mathbb{R}} f(x) \phi_n(x) \, dx = f(0).$$

(b) Let $\{B_n\}$ be a countable collection of Lebesgue measurable subsets of $\mathbb{R}$. Assume that

$$\sum_{n=1}^{\infty} \mu(B_n) < \infty,$$

where μ denotes Lebesgue measure. Prove that

$$\mu\left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} B_k\right) = 0.$$

(6) Let N be a fixed positive integer. Let $\{p_n(x)\}$ be the sequence of polynomials of degree N given by

$$p_n(x) = a_N^{(n)} x^N + a_{N-1}^{(n)} x^{N-1} + \cdots + a_1^{(n)} x + a_0^{(n)}.$$

(a) If each of the numerical sequences $\{a_k^{(n)}\}$ converge, that is,

$$\lim_{n\to\infty} a_k^{(n)} = a_k, \quad 0 \le k \le N,$$

prove that the sequence $\{p_n(x)\}$ converges uniformly to $p(x)$ on the unit interval $[0,1]$ where

$$p(x) = a_N x^N + a_{N-1} x^{N-1} + \cdots + a_1 x + a_0.$$

(b) If the sequence $\{p_n(x)\}$ converges uniformly on $[0,1]$ to a function $f(x)$, prove that $f(x)$ must also be a polynomial whose degree is at most N.

2. Linear Algebra

Remember: you are to answer 2 out of the following 3 Linear Algebra problems.

---

(1) Let A be a $n \times n$ complex matrix so that $A^2 = I$. Show that the following statements are equivalent:
   (a) A is Hermitian;
   (b) A is normal;
   (c) A is unitary;
   (d) all the singular values of A are equal to 1.

(2) Let $A = \begin{pmatrix} 2 & 1 & 0 \\ -1 & 4 & 0 \\ -1 & 3 & 3 \end{pmatrix}$.
   (a) Find the eigenvalues of A.
   (b) Find the eigenvectors of A.
   (c) Determine the Jordan canonical form of A.
   (d) Give the minimal polynomial of A.

(3) Let A, B, and $A - B$ be matrices with positive entries.
   (a) If x is a vector with positive entries, show that $Ax > Bx$; (that is, $Ax - Bx$ has only positive entries).
   (b) Show that $\rho(A) > \rho(B)$, where ρ is the spectral radius. (Hint: let x be a Perron vector of A).
   (c) For $A = \frac{1}{9}\begin{pmatrix} 2 & 2 & 4 \\ 1 & 1 & 2 \\ 2 & 2 & 4 \end{pmatrix}$, compute $\lim_{m\to\infty} A^m$.