DEPARTMENT OF MATHEMATICS
DREXEL UNIVERSITY
PH.D. QUALIFYING EXAMINATION
SEPTEMBER 18, 2009

Instructions:

- The exam consists of six problems which are equally weighted.
  The time for examination is $3\frac{1}{2}$ hours.

- In the Analysis section, do two problems out of (1),(2),(3) and do two out of (4),(5),(6) for a total of four problems.

- Do **2** out of 3 of the linear algebra questions in Section 2.

- Indicate clearly which of your questions are to be graded. If you do **not** indicate which of your questions are to be graded, the default will be to grade questions one through four of the analysis section and questions one and two of the linear algebra section.

- Please ask the proctor about any obvious typographic errors.

- Along with this list of problems, you will be given two examination notebooks. Use one of them for presenting your solutions. The other one may be used for auxiliary calculations. Both notebooks must be submitted when the exam is over.

- Every solution should be given a concise but sufficient explanation and written up legibly. This is a closed book exam.

- No electronic devices are allowed.

Remember: do two problems out of (1),(2),(3) and do two out of (4),(5),(6) for a total of four problems in this section.

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(1) (a) Prove the interior extremum theorem: Let $f : [a,b] \to \mathbb{R}$, and let $c \in (a,b)$. Assume f has a relative extremum at c. If $f'(c)$ exists, then $f'(c) = 0$.

(b) Prove Rolle's Theorem: Let $f : [a,b] \to \mathbb{R}$ be continuous, and assume f is differentiable on (a,b). Assume $f(a) = f(b)$. Then there exists $c \in (a,b)$ such that $f'(c) = 0$.

(2) (a) Consider the non-linear system of equations, where a and b are constants.

$$\begin{aligned} x+y &= a, \\ xy - x &= b. \end{aligned}$$

Find the set of $(a,b) \in \mathbb{R}^2$ for which this system has a solution $(x,y) \in \mathbb{R}^2$ by solving for x and y in terms of a and b. In other words, describe the range of the map $F : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$F(x,y) = (x+y, xy - x).$$

Just to be clear about what form the set of range points should be in, you should be able to easily draw a picture of the range. Drawing the picture is not required.

(b) Note that $F(0,1) = (1,0)$, that is, the above system has at least one solution for $a = 1$, $b = 0$. Show that $F'(0,1)$ is singular. Does this contradict the inverse function theorem? Explain.

(3) (a) Show that the function

$$f(x) = \begin{cases} x^2 \sin(\frac{1}{x}), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

is differentiable at $x = 0$.

(b) Show that the function in (a) is not continuously differentiable at $x = 0$.

(4) Let $f \in C^2[a,b]$, the space of all functions on $[a,b]$ with a continuous second derivative. Let $x_n \in [a,b]$ be a sequence such that $x_n \to y$, and $x_n \neq y$ for any n. Assume that $f(x_n) = 0$, for all n.
(a) Show that $f'(y) = 0$.
(b) Show that $f''(y) = 0$.

(5) In this problem, all integrals are understood to be Lebesgue integrals. Let H_n and H be given by the following:

$$H_n(x) = \begin{cases} 0, & x \leq 0, \\ nx, & x \in (0, \frac{1}{n}), \\ 1, & x > \frac{1}{n}, \end{cases} \qquad H(x) = \begin{cases} 0, & x \leq 0, \\ 1, & x > 0. \end{cases}$$

(a) Show that H_n does not converge uniformly to H, but that $\int_{-1}^{1} H_n(x)\, dx$ does converge to

$$\int_{-1}^{1} H(x)\, dx.$$

(b) Let $u : [-1, 1] \to \mathbb{R}$ be a continuous function. Show that $\int_{-1}^{1} H_n(u(x)) \, dx$ converges to $\int_{-1}^{1} H(u(x)) \, dx$.

(6) Let $C^1(X)$ be the space of all functions on a set X with a continuous first derivative. Let $f_n \in C^1(X)$ for some set X, for all n. Assume there exists $M > 0$ such that $\|f_n\|_\infty + \|f_n'\|_\infty < M$ for $n = 1, 2, \ldots$. Recall the notation $\|g\|_\infty = \sup\{|g(x)| : x \in X\}$.
   (a) If $X = [0, 1]$, show that there exists a subsequence f_{n_k} which converges uniformly to a function $f \in C[0, 1]$.
   (b) If $X = [0, \infty)$, give an example of such a sequence f_n, which has no uniformly convergent subsequence.

2. LINEAR ALGEBRA

Remember: you are to answer 2 out of the following 3 Linear Algebra problems.

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(1) (a) Let X be a real or complex n-dimensional vector space, and let $e_1, \ldots, e_m$ be vectors in X. Prove that if $m \geq n + 2$ then there exist numbers $a_1, \ldots, a_m$ not all equal to zero such that $\sum_{j=1}^{m} a_j e_j = 0$ and $\sum_{j=1}^{m} a_j = 0$.

   (b) Prove that if U is a unitary matrix and S is a Hermitian positive semi-definite matrix then $|\text{trace}(US)| \leq \text{trace}\, S$.

   (c) Let $s_1, \ldots, s_n$ be the singular values of a $n \times n$ matrix A. Prove that the eigenvalues of $\begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix}$ are equal to $s_1, \ldots, s_n, -s_1, \ldots, -s_n$.

(2) Find the Jordan canonical form (up to a permutation of diagonal blocks) of the matrix A if

   (a) $A = \begin{bmatrix} 0 & -4 & 0 \\ 1 & -4 & 0 \\ 1 & -2 & -2 \end{bmatrix}$.

   (b) $A = \begin{bmatrix} \alpha & 0 & 1 & 0 & \cdots & 0 \\ 0 & \alpha & 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 0 & 1 \\ \vdots & & & \ddots & \alpha & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & \alpha \end{bmatrix}$, where A is square of size $n \geq 3$, and α is a scalar.

(3) (a) Prove that if at least one of matrices A, B is non-singular then AB and BA are similar.

   (b) Find all square matrices which are similar to themselves only. Prove your answer.

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