DEPARTMENT OF MATHEMATICS
DREXEL UNIVERSITY
PH.D. QUALIFYING EXAMINATION
SEPTEMBER 16, 2008

Instructions:

- The exam consists of six problems which are equally weighted.
  The time for examination is $\mathbf{3\frac{1}{2}}$ hours.
- Do **4** out of 6 of the analysis questions in Section 1.
- Do **2** out of 3 of the linear algebra questions in Section 2.
- Indicate clearly which of your questions are to be graded. If you do **not** indicate which of your questions are to be graded, the default will be to grade questions one through four of the analysis section and questions one and two of the linear algebra section.
- Please ask the proctor about any obvious typographic errors.
- Along with this list of problems, you will be given two examination notebooks. Use one of them for presenting your solutions. The other one may be used for auxiliary calculations. Both notebooks must be submitted when the exam is over.
- Every solution should be given a concise but sufficient explanation and written up legibly. Try to keep a one inch margin on the papers.
- This is a closed book exam.
- No electronic devices are allowed.

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1. ANALYSIS

Remember: you are to answer 4 out of the following 6 Analysis problems.

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(1) Let $f(x)$ be uniformly continuous on $(0,\infty)$.
(a) Prove that the limit $\lim_{x\to 0^+} f(x)$ exists.
(b) Prove or provide a counterexample for the claim that the $\lim_{x\to\infty} f(x)$ exists.

(2) Let X and Y be metric spaces. Let $f : X \to Y$. Suppose that $X = \bigcup_{n=1}^{\infty} U_n$, and that f is continuous on each set U_n.
(a) If each U_n is an open subset of X, prove that f is continuous on X.
(b) Show that f may fail to be continuous on all of X if we assume instead that each U_n is a closed set.

(3) (a) Find the limit

$$\lim_{n\to\infty}\left[\int_0^{\pi/2}\left(\frac{\sin x}{x}\right)^n dx\right]^{1/n}.$$

Justify your reasoning. *Hint*: you may use the fact that $\frac{\sin x}{x}$ is decreasing on $[0,\pi/2]$.
(b) Prove that there is NO sequence of polynomials $\{p_n(x)\}$ that converge uniformly to $\sin x$ on the infinite closed interval $[0,\infty)$.

(4) Let $f : \mathbb{R} \to \mathbb{R}$ be infinitely differentiable. Let $f^{(n)}(x)$ denote its n-th derivative. Assume that $|f^{(n)}(x) - f^{(n-1)}(x)| < 2^{-n}$ for all $x \in \mathbb{R}$.
(a) Prove that the sequence $\{f^{(n)}(x)\}$ converges uniformly on $\mathbb{R}$.
(b) Prove that

$$\lim_{n\to\infty} f^{(n)}(x) = ce^x,$$

for some constant c.

(5) (a) Evaluate the limit $\displaystyle\lim_{n\to\infty}\int_0^n\left(1+\frac{x}{n}\right)^n e^{-2x}\,dx$. Justify your reasoning.
(b) If $f : [0,1] \to \mathbb{R}$ is continuous, prove that $\lim_{n\to\infty}\int_0^1 f(x^n)\,dx = f(0)$.

(6) (a) Is there a continuously differentiable map T of $\mathbb{R}^2$ back into itself such that its differential at the point (x,y) is

$$T'(x,y) = \begin{bmatrix} 2x^2y & x^3 \\ y & x \end{bmatrix}.$$

Prove your answer.
(b) Let $F : \mathbb{R}^2 \to \mathbb{R}$ be continuously differentiable such that $F(0,0) = 0$. Find conditions on the function F that guarantee that the equation

$$F(F(x,y),y) = 0$$

can be solved for y as a function of x near the point $(0,0)$.

Remember: you are to answer 2 out of the following 3 Linear Algebra problems.

(1) Let $A = (a_{ij})_{i,j=1}^n$ be a $n \times n$ normal matrix, and let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of A. Let $H = AA^* + 5A$ where A^* is the complex conjugate transpose of A.
   (a) Express the eigenvalues of H in terms of those of A, and explain your answer.
   (b) Show that $\sum_{i=1}^n \sum_{j=1}^n |a_{ij}|^2 = \sum_{i=1}^n |\lambda_i|^2$.
   (c) Does there exists a non normal matrix that satisfies the equality in (b)? If so, provide one. If not, explain why not.

(2) (a) Let $A \in \mathbb{R}^{6\times 6}$ with characteristic polynomial $p(x) = (x+2)^4(x-1)^2$ and minimal polynomial $m(x) = (x+2)^2(x-1)$. What are the possible Jordan canonical form(s) for A (up to permutation of Jordan blocks)?
   (b) Suppose $A \in \mathbb{C}^{n\times n}$ and $x \in \mathbb{C}^n$ satisfy $A^k x \neq 0$ and $A^{k+1}x = 0$. Prove that $\{x, Ax, \ldots, A^k x\}$ is linearly independent.

(3) (a) Let $A \in \mathbb{C}^{n\times n}$. Prove that A is invertible if and only if there is a polynomial $p \in \mathbb{C}[x]$, with zero constant term, such that $p(A) = I_n$ (the $n \times n$ identity matrix).
   (b) Show that if A is similar to a unitary matrix, then A^* and A^{-1} are similar.
   (c) Does there exists a $n \times n$ matrix ($n \geq 2$) with positive entries that is similar to a unitary matrix?