

DEPARTMENT OF MATHEMATICS  
DREXEL UNIVERSITY  
PH.D. QUALIFYING EXAMINATION

June 28, 2006

**About this exam:**

Along with this list of problems, you will be given two examination notebooks. Use one of them for presenting your solutions. The other one may be used for auxiliary calculations. Both notebooks must be submitted when the exam is over. Every solution should be given a concise but sufficient explanation and written up legibly. This is a closed book exam. No electronic devices are allowed. The exam consists of seven problems (100 points) and one extra credit problem (15 points).

1. **a. (8 points)** Let  $f$  be a bounded increasing function on  $[0, 1]$ . Prove that  $f$  has at most countably many points of discontinuity.

**b. (7 points)** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be continuous for all  $x \in \mathbb{R}$  and differentiable for all non-zero  $x$ . Prove that if  $\lim_{x \rightarrow 0} f'(x) = 0$ ,  $f$  must be differentiable at 0.

2. Prove the statements in **a.-c.** if they are correct or provide a counterexample otherwise.

**a. (5 points)** Let  $f$  be a Riemann integrable function on  $[a, b]$  such that  $f(x) = 0$  for any  $x \in [a, b] \cap \mathbb{Q}$ . Then  $\int_a^b f(x) dx = 0$ .

**b. (5 points)** Let  $f$  be a Lebesgue integrable function on  $[a, b]$  such that  $f(x) = 0$  for any  $x \in [a, b] \cap \mathbb{Q}$ . Then  $\int_a^b f(x) dx = 0$ .

**c. (10 points)** Let a nonnegative function  $f$  be Lebesgue integrable on  $[a, b]$  and such that  $\int_a^b f(x) dx = 0$ . Then  $f = 0$  almost everywhere on  $[a, b]$ .

3. **a. (8 points)** Let  $a_j \geq 0$  such that  $\sum_{j=1}^n a_j < \ln(n)$  for all  $n \in \mathbb{N}$ . Prove that the series

$\sum_{j=1}^{\infty} a_j/j$  converges.

**b. (8 points)** Evaluate  $\lim_{n \rightarrow \infty} \int_0^{\infty} \frac{\sin(x/n)}{(1+x/n)^n} dx$ . Justify all steps.

**c. (6 points)** Let  $f$  be a continuous strictly positive function on  $[0, 1]$ . Show that for each positive integer  $n$ , there is  $\theta_n \in [0, 1]$  such that

$$\int_0^{\theta_n} f(x) dx = \frac{1}{n} \int_0^1 f(x) dx.$$

Find  $\lim_{n \rightarrow \infty} n\theta_n$ . Justify all steps.

4. **(8 points)** Let  $\lim_{n \rightarrow \infty} a_n = a$ . Prove

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n |a_j - a| = 0.$$

5. Recall that the polar form of a  $m \times n$  matrix  $A$  is given by  $A = PU$ , where  $P$  is positive semidefinite with  $\text{rank } P = \text{rank } A$ , and  $U$  has orthonormal rows (that is,  $UU^* = I$ ).

**a. (6 points)** Let  $A$  be a  $n \times n$  invertible matrix. Prove that  $A$  has a polar form. (Do not quote Theorem 7.3.2 from the book; Hint: let  $P = (AA^*)^{1/2}$ .)

**b. (5 points)** Let  $A$  be invertible. Show that  $A$  is normal (i.e.,  $AA^* = A^*A$ ) if and only if  $P$  and  $U$  commute, where  $A = PU$  is a polar form of  $A$ .

**c. (6 points)** Compute the polar form of

$$A = \begin{pmatrix} 3 & 1 \\ 3 & -1 \end{pmatrix}.$$

6. Let  $a_i \in \mathbb{R}$  and  $b_i > 0$ ,  $i \in \mathbb{N}$ . Consider the matrices

$$A_n = \begin{pmatrix} a_1 & 1 & 0 & \cdots \\ b_1 & a_2 & 1 & \ddots \\ \cdots & \ddots & \ddots & \ddots \\ \cdots & 0 & b_{n-1} & a_n \end{pmatrix}.$$

**a. (6 points)** Show that  $A_n$  is similar to

$$B_n = \begin{pmatrix} a_1 & \sqrt{b_1} & 0 & \cdots \\ \sqrt{b_1} & a_2 & \sqrt{b_2} & \ddots \\ \cdots & \ddots & \ddots & \ddots \\ \cdots & 0 & \sqrt{b_{n-1}} & a_n \end{pmatrix}.$$

(Hint: use a diagonal similarity.)

**b. (6 points)** Show that the eigenvalues of  $A_{n-1}$  and  $A_n$  interlace. (Hint: use part (a).)

**c. (6 points)** Assume that  $\det A_n > 0$  for all  $n \geq 1$ . Show that the eigenvalues of  $A_n$  are all positive.

**7. Extra credit**

**a. (7 points)** Let  $A$  and  $B$  be  $n \times n$  positive definite matrices. Show that for  $x > 0$  the second derivative of  $f(x) = \log \det (A + xB)$  is always negative.

**b. (8 points)** Let  $A$ ,  $B$  and  $C$  be linearly independent positive definite matrices. Show that the Hessian for the function  $f(x, y) = \log \det (A + xB + yC)$  is negative definite for  $x, y > 0$ .